AN AUTOMATA-THEORETIC APPROACH TO THE STUDY OF THE INTERSECTION OF TWO SUBMONOIDS OF A FREE MONOID

LAURA GIAMBRUNO AND ANTONIO RESTIVO

Abstract. We investigate the intersection of two finitely generated submonoids of the free monoid on a finite alphabet. To this end, we consider automata that recognize such submonoids and we study the product automata recognizing their intersection. We start by proving a result of Karhumaki on the characterization of the intersection of two submonoids of rank two, in the case of prefix (or suffix) generators. In a more general setting, for an arbitrary number of generators, we prove that if H and K are two finitely generated submonoids generated by prefix sets such that the product automaton associated to $H \cap K$ has a given special property then $\widetilde{rk}(H \cap K) \leq \widetilde{rk}(H) \cdot \widetilde{rk}(K)$ where $\widetilde{rk}(L) = max(0, rk(L) - 1)$ for any free submonoid L.

The starting motivation of this paper was to study the intersection of two submonoids generated by two elements of the free monoid over a finite alphabet A.

Given a finite alphabet A, let A^* be the free monoid on A. The intersection of two submonoids of the free monoid A^* was first studied by Tilson ([7]) who proved that the intersection of two free submonoids of A^* is still free.

Karhumaki in 1984 ([4]) gave a characterization of the intersection of two submonoids generated by two elements. He proved that given two submonoids H and K of A^*, if both H and K are of rank two, then $H \cap K$ is a monoid generated by at most two words or by a regular language of a special form. In particular if H and K are generated by prefix (or suffix) sets of two words, and $H \cap K$ is not finitely generated, then this intersection has the form $(\alpha\beta^*\gamma)^*$ where $\alpha, \beta, \gamma \in A^*$.

In this paper, in particular, we prove the result of Karhumaki in the case of two submonoids generated by prefix (or suffix) sets using a more intuitive approach.

When dealing with the intersection of two submonoids of rank two it is natural to refer to a more general problem in the theory of free groups known as the 'Hanna Neumann conjecture'. This conjecture deals with the problem of finding an upper bound of the rank of the intersection of two finitely generated subgroups.

In 1956 Hanna Neumann ([5]) proved that if H and K are two subgroups of finite rank then $\widetilde{rk}(H \cap K) \leq 2\widetilde{rk}(H)\widetilde{rk}(K)$ where for a free group T,

$\widetilde{rk}(T) = max(rk(T) - 1, 0)$ with $rk(T)$ the rank of T. Then she made the following conjecture, known nowadays as the 'Hanna Neumann conjecture':

$$\widetilde{rk}(H \cap K) \leq \widetilde{rk}(H)\widetilde{rk}(K)$$

In 1991 Walter Neumann ([6]) formulated a stronger conjecture known as 'Strengthened Hanna Neumann conjecture' (in short SHN) and in 2002 Meakin and Weil ([3]) proved that SHN holds for the class of positively generated subgroups of the free group $F(A)$ on A finite alphabet that are generated by words in A^*. This last result suggested us to propose the problem of Hanna Neumann for finitely generated submonoids of a free monoid, in the case that their intersection is finitely generated.

Some of the basic tools in dealing with the Hanna Neumann conjecture for free groups makes use of the representation of subgroups of the free group by graphs (or automata). The same tools are still available when dealing with the intersection of two submonoids of the free monoid. For this purpose we refer to the well known correspondence (see [1]) between submonoids on the free monoid on a finite alphabet A and automata on A.

Through the study of the product of two automata associated to two finitely generated submonoids H and K, we were able to prove that if H and K are submonoids generated by prefix sets such that the product automaton associated to $H \cap K$ has a given special property then $\widetilde{rk}(H \cap K) \leq \widetilde{rk}(H)\widetilde{rk}(K)$.

In the general case we have found a family of examples such that $rk(H \cap K) = 2^{log_2(rk(H))log_2(rk(K))}$. We conjecture that this is the worst case.

Moreover if the two submonoids H and K are generated by prefix (or suffix) sets of two elements and if $H \cap K$ is not finitely generated then this intersection has the form $(\alpha\beta^*\gamma)^*$ where $\alpha, \beta, \gamma \in A^*$, that is the result of Karhumaki.

In the first part of the paper we describe briefly the correspondence between submonoids of the free monoid on a finite alphabet and the class of automata with one final state equal to the initial one.

We treat with final states automata on a finite alphabet A. Let A be an alphabet. An *automaton* over A $\mathcal{A} = (Q, I, T, \mathcal{F})$ consists of a finite set Q of states, of two subsets I and T of Q called sets of initial and final states, respectively, and of a set $\mathcal{F} \subset Q \times A \times Q$ whose elements are called edges. A path in $\mathcal{A}$ is a finite sequence $p = p_1p_2 \dots p_n$ of consecutive edges $p_i = (x_i, a_i, y_i)$ (i.e. such that $y_i = x_{i+1}$ for $1 \leq i \leq n-1$). We say that a path c is a cycle in x if it starts and ends at x. A cycle c in x is simple if it is not the null path and if no interior state is equal to x.

We can think of $\mathcal{A}$ as a graph whose set of vertices is the set of states Q and the set of labelled edges is $\mathcal{F}$.

Given a graph we say that a vertex v is a *branch point* (in short bp) if the degree of v (i.e. the number of edges incident to v) is greater than two. We say that a vertex v is a *branch point going out* (in short bpo) if v is a branch point and if the number of edges going out is at least two and we say that v is a *branch point going in* (in short bpi) if v is a branch point and if the number of edges coming in is at least two.

An automaton $\mathcal{A}$ over A is a *deterministic automaton* if $card(I) = 1$ and if for each state x and for each $a \in A$ there is at most an edge starting in x with label a.

Given an automaton $\mathcal{A}$ on a finite alphabet A, we say that it is *trim* if all the states of the automaton are accessible and coaccessible. We say that $\mathcal{A}$ is a *monoidal automaton* if it is a trim automaton with a unique final state equal to the initial one.

By [1] we have that if $\mathcal{A}$ is a monoidal automaton on A recognizing H submonoid of A^*, then H is free with basis the set of labels of the simple cycles in the initial-final state 1 if and only if $\mathcal{A}$ is an unambiguos automaton.

To each submonoid H generated by a finite set X_H is associated $\mathcal{F}_{X_H}$ the *flower automaton* of X_H. Such automaton is a monoidal automaton such that all the cycles visit the initial-final state 1, intersect themselves only in 1 and the cycles in 1 that visit just twice 1 have as labels the words of X_H. We have $L(\mathcal{F}_{X_H})=H$. Conversely to a monoidal automaton $\mathcal{A}$ is associated the submonoid $H = L(\mathcal{A})$ of A^*. We remark that the flower automaton associated to a submonoid is not necessarily deterministic.

We say that $\mathcal{A}$ is a *semi-flower automaton* if it is a monoidal automaton such that all the cycles visit the unique initial-final state 1. Hence in a semi-flower automaton the cycles in the initial-final state 1 intersect themselves not necessarily only in 1.

We have that if $\mathcal{A}$ is an unambiguos monoidal automaton on A recognizing H submonoid of A^*, then H is finitely generated if and only if $\mathcal{A}$ is a semi-flower automaton.

We say that $\mathcal{A}$ is a *strongly semi-flower automaton* if it is a semi-flower automaton such that there are not bpi different from 1.

In this setting we have the following results:

Theorem 1. *If $\mathcal{A}$ is a strongly semi-flower automaton with v vertices and e edges and $H = L(\mathcal{A})$ then $rk(H) \leq e - v + 1$.*

And if we consider unambiguos automata we get the following:

Theorem 2. *If $\mathcal{A}$ is an unambiguos strongly semi-flower automaton with v vertices and e edges and $H = L(\mathcal{A})$ then $rk(H) = e - v + 1$.*

We remark that a similar result holds for free groups: if $\mathcal{A}$ is an inverse automaton with v vertices and e edges recognizing a subgroup H then $rk(H) = e - v + 1$.

In the setting of a binary alphabet A_2 the following holds: if $\mathcal{A}$ is a deterministic trim automaton with non empty language and with v vertices and e edges on A_2 then $e - v = \sharp\, bpo$ (i.e. $e - v$ is the number of bpo).

Given a submonoid H generated by a prefix finite set we can associated to it an automaton recognizing H in the following way. Let $U = \{u_1, \ldots, u_n\} \subseteq$

A^* be a finite prefix set. Let $P(U)$ be the set of the proper prefixes of elements of U together with ε, the empty word in A^*. Let $\mathcal{A}_U = (Q, \{\varepsilon\}, \{\varepsilon\}, \delta)$ with $Q = P(U)$ and for each $u \in P(U)$ for each $a \in A$ $\delta(u, a) = ua$ if $ua \notin U$, $\delta(u, a) = \varepsilon$ if $ua \in U$.

It is proved the following proposition:

Proposition 3. *Let U be a finite prefix set then $\mathcal{A}_U$ is a deterministic strongly semi-flower automaton.*

If $\mathcal{A}$ is a deterministic strongly semi-flower automaton recognizing the submonoid $H = X^*$ then X is a finite prefix set.

So we have that if $H = X^*$ then X is a finite prefix set if, and only if there exists a strongly semi-flower automaton recognizing it.

From now on, given a submonoid H generated by a finite prefix set X, let's denote by $\mathcal{A}_H$ the automaton $\mathcal{A}_X$ as before associated to X.

Resuming all the precedent results we get in the setting of a binary alphabet the following theorem:

Theorem 4. *If H is a submonoid of ${A_2}^*$ finitely generated by a prefix set, then $\mathcal{A}_H$ is a deterministic strongly semi-flower automaton and $\widetilde{rk}(H) = \sharp$ bpo.*

This property of prefix codes allowed us to prove our result in the prefix case.

In the second part of the paper we investigate the intersection of two finitely generated submonoids of the free monoid on a binary alphabet A by studying the product of two automata associated to them. All the results obtained are then extended to an arbitrary finite alphabet.

Given $\mathcal{A}_1$ and $\mathcal{A}_2$ two monoidal automata their product is still monoidal and recognizes $L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$. Moreover it is well known that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are two deterministic automata then the product is still a deterministic automaton. Instead the product of two trim automata is not necessarily a trim automaton. Moreover we have that the product of two semi-flower automata is not necessarily a semi-flower automaton and that the product of two strongly semi-flower automata if it is a semi-flower automaton then it is not necessarily a strongly semi-flower automaton.

Given H and K submonoids finitely generated by prefix sets, the corresponding $\mathcal{A}_H$ and $\mathcal{A}_K$ are deterministic monoidal automata, and so is $\mathcal{A}_H \times \mathcal{A}_K$.

Let consider in $\mathcal{A}_H \times \mathcal{A}_K$ only the set of accessible and coaccessible states. We have that $H \cap K$ is finitely generated if and only if $\mathcal{A}_H \times \mathcal{A}_K$ is a semi-flower automaton. If $\mathcal{A}_H \times \mathcal{A}_K$ is a deterministic strongly semi-flower automaton then by studying the nature of the bpo in $\mathcal{A}_H \times \mathcal{A}_K$ we prove the following

Theorem 5. *Let H and K be submonoids finitely generated by prefix sets such that $H \cap K$ is finitely generated. If $\mathcal{A}_H \times \mathcal{A}_K$ is a strongly semi-flower automaton then $\widetilde{rk}(H \cap K) \leq \widetilde{rk}(H)\widetilde{rk}(K)$.*

If H and K are submonoids finitely generated by prefix sets such that $H \cap K$ is finitely generated, if $\mathcal{A}_H \times \mathcal{A}_K$ is a semi-flower automaton not strongly then it is not more true that $\widetilde{rk}(H \cap K) \leq \widetilde{rk}(H)\widetilde{rk}(K)$. In fact there is a family of examples such that $rk(H \cap K) = 2^{log_2(rk(H))log_2(rk(K))}$:

Example 6. *Let p and q be two positive coprime integers. Let A be a binary alphabet and let $H = A^p$ and $K = A^q$ (A^p is the set of words in A^* of length p). It is $rk(H) = 2^p$ and so $p = log_2(rk(H))$. It is $H \cap K = A^{pq}$ and $rk(H \cap K) = 2^{pq} = 2^{log_2(rk(H))log_2(rk(K))}$.*

We conjecture that this is the worst case.

If H and K are generated by prefix sets of two elements we get the result of Karhumaki:

Theorem 7. *If H and K are submonoids finitely generated by prefix (or suffix) sets of two elements and $H \cap K$ is finitely generated then $rk(H \cap K)$ is at most two. If $H \cap K$ is not finitely generated then there exist $\alpha, \beta, \gamma \in A^*$ such that $H \cap K = (\alpha(\beta)^*\gamma)^*$.*

The case when H and K are generated by suffix sets can be easily reduced to the prefix case.

References

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Dipartimento di Matematica e Applicazioni, Università di Palermo, Via Archirafi 34, 90123 Palermo, Italy

E-mail address: lgiambr@math.unipa.it

Dipartimento di Matematica e Applicazioni, Università di Palermo, Via Archirafi 34, 90123 Palermo, Italy

E-mail address: restivo@math.unipa.it